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LETTER TO THE EDITOR

Numerical study of solitons in the damped AC-driven Toda lattice

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Abstract. It is demonstrated using numerical experiments that the damped AC-driven Toda lattice can support stable propagation of solitons, if the drive strength is above a certain threshold value, and the drive frequency and phase are properly chosen. This phenomenon is thoroughly studied by varying the dissipation factor and the drive parameters. In particular the threshold values of the drive parameter corresponding to different soliton velocities are found.

Nonlinear models of contemporary condensed matter physics demonstrate a rich variety of dynamic behaviour. In many cases, these models support stable propagation of solitons, which are localized collective excitations in the corresponding physical systems. The solitons are well known to demonstrate remarkable dynamic properties: stable propagation, elastic collisions, etc. However, this nice picture of soliton dynamics is inevitably altered by dissipation, which gives rise to a gradual attenuation of the solitary pulses. To compensate the dissipative losses and thus to stabilize the propagation of the solitons, one should apply some external drive to the system. A well known example of this is a long Josephson junction, where weak dissipative losses, caused by tunnelling of normal electrons across the dielectric barrier, can be compensated by a uniformly distributed DC bias current. This mechanism provides for the stable propagation of the solitons in the form of magnetic flux quanta (fluxons), the fluxons move in the junction at an equilibrium velocity determined by the balance between the dissipation and the DC-drive [1]. This DC-driven motion of solitons is a generic effect and using it one can analyse, e.g., the propagation of phase solitons in a damped charge-density-wave system driven by a DC electric field, the motion of domain walls in a damped ferromagnet driven by a DC magnetic field and so on (see the review papers [2, 3]).

While the models of such systems as Josephson junctions and charge density waves are continuum, a more precise description of solitary wave propagation in molecular chains and in strings of absorbed atoms should be discrete. Discrete lattice models can also support solitonic excitations, but then the dynamics of solitons drastically differ from that in the continuum models. For example, the presence of an exact one-soliton solution to a continuum model is a rather generic property and it does not imply their exact integrability, while, in the discrete case only, exactly integrable models admit undistorted propagation of a soliton. In the general (non-integrable) case, a

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solitary excitation must spread out due to radiative losses (although simulations demonstrate that sometimes these losses may be extremely small, rendering the propagation of a soliton in a non-integrable lattice almost stable [2, 3]). However, in real discrete systems, such as atomic chains, etc, a much more important factor is the dissipative damping. So, to support propagation of solitons in these chains, one needs to design a drive that can compensate the radiative and dissipative losses.

Recently it has been suggested [4] that there is a new mechanism to sustain soliton propagation in discrete dissipative systems, which has no analogue in the continuum models: AC (time-periodic) drive can compensate dissipative and/or radiative losses in a chain. The physical idea underlying this mechanism is that, if the chain's spacing is d, and the AC-drive's frequency is ω, they give rise to the velocities

$$v_N = \omega d/2\pi N \qquad N = \pm 1, \pm 2, \pm 3, \ldots \tag{1}$$

at which a resonance between the periodic process of passage of the soliton through sites of the lattice (chain) and the periodic AC-drive takes place. In [4] it has been shown that the regimes of propagation of the soliton with resonance velocities given by (1) are stable under certain conditions. The necessary condition for these regimes to occur is that the drive's amplitude e should exceed a certain threshold value e_{thr} proportional to the dissipative constant a[4]. The value e_{thr} is the minimum one at which the AC-drive can compensate the dissipation; at $e > e_{\mathrm{thr}}$ the exact compensation is provided by an expedient phaseshift between the soliton's law of motion and the AC-drive.

The aim of the present work is to demonstrate by direct numerical simulations the possibility of the AC-drive propagation of a soliton in a damped chain. As the paradigm of soliton-supporting chains, we will take the Toda lattice (TL), which is well known to be exactly integrable in the absence of dissipation and drive [5]. Dynamics of solitons in the damped TL (without drive) have been studied numerically in several works [6–9].

Following [4] we will take the AC-driven damped TL in the following form†:

$$\partial_t^2 \ln(1+V_n) + a\partial_t \ln(1+V_n) = V_{n+1} + V_{n-1} - 2V_n + 2(-1)^n e\cos(\omega t) \tag{2}$$

where, as defined above, a, e and ω are the dissipative constant, the drive amplitude, and frequency. The spin-changing factor $(-1)^n$ is necessary to provide for real input of energy from the AC-drive (as proposed in [4] one may regard this model as a lattice of particles with alternating charges placed in an external field). In this case the driving frequency must be [4]

$$\omega = (2N+1)\pi v \qquad N = 0, 1, 2, \ldots \tag{3}$$

and the corresponding threshold value for the drive parameter can be estimated as

$$e_{\mathrm{thr}} \approx 4a\omega \ln(v) \tag{4}$$

for sufficiently large v [4].

We have numerically integrated equation (2) using the Bulirsch–Stoer algorithm as an integration method with periodic boundary conditions [10]. The total number of lattice points was 400. As an initial state we have used the ideal one-soliton solution $V_n(0) = \sinh^2(\Omega)\,\mathrm{sech}^2(\Omega(n-n_0))$ with the initial position $n_0 = 10$.

† Equation (2) is equivalent to equation (1) from [4] (for the TL model) if one substitutes $\ln(1+V_n) = y_{n-1} - y_n$.

In order to examine the evolution of the soliton under external drive we have solved numerically the eigenvalue problem of the Toda lattice [5] at the moments of time $t = 0, 10, \ldots, 150$. With this method it is possible to get quite accurate information about the temporary soliton content of the lattice. (By looking at the form of the soliton solution we cannot say much about its amplitude, especially when it is narrow, because its maximum will not be exactly at the lattice point. The eigenvalue method is much more accurate than fitting the soliton solution to the values of the lattice points in order to find the real amplitude.)

The amplitude $A = \sinh^2(\Omega)$ of the soliton corresponding to the dominant eigenvalue (other eigenvalues represent the tail and other small perturbations) is shown in figure 1 as a function of time for different values of the drive parameter e. The initial amplitude was 24.37 ($\Omega = 2.30$), the velocity $v = 2.14$, the dissipation factor $a = 0.002$ and the first resonance frequency ($N = 0$) $\omega = \pi v = 6.774\,34$. There is clear threshold value at $e = 0.0490 \pm 0.0005$: with smaller values the amplitude decays exponentially but with higher values the mean amplitude is constant. With other initial amplitudes (velocities) there is a similar threshold value for e.

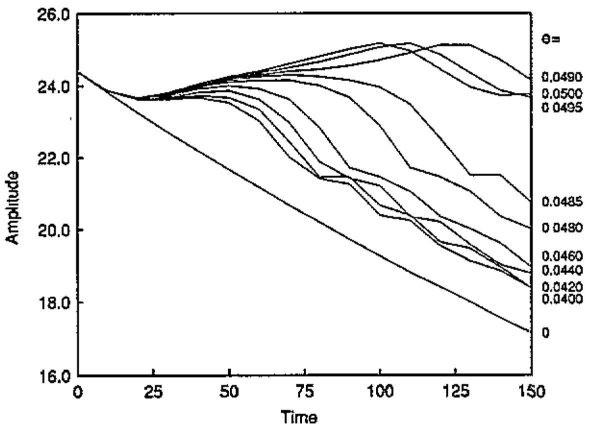

Figure 1. The amplitude of the soliton as a function of time with different values of the drive parameter e. The initial amplitude was 24.37 ($\omega = 2.3$) and the dissipation factor $a = 0.002$.

Figures 2(a) and 2(b) show explicitly the time evolution of a solution by giving snapshots of the lattice at various times for two different drives. The total number of lattice points was 1000 and $A = 24.37$, $\Omega = 2.30$, $a = 0.01$. In figure 2(a) the drive $e = 0.23$ is below the threshold value and the amplitude of the soliton decreases, but with $e = 0.24$ in figure 2(b) the drive is above the threshold value and the mean amplitude of the soliton does not decrease. Since the soliton is very narrow (it is localized within a couple of lattice points) its maximum does not often fall on the lattice point at the chosen times and therefore its apparent amplitude fluctuates strongly.

The effect of further increasing the drive e is presented in figure 3 where in all cases e is well above the threshold value. With increasing drive the period of oscillation in the amplitude of the soliton get smaller. It is natural to expect that the period approaches infinity as the drive parameter approaches the threshold value. It should be noted that only the frequency of the oscillation changes with e, while the average amplitude and the amplitude of the oscillation stay the same.

The phase between the soliton and the external drive is important for the resonance condition. In figure 4 the amplitude is shown as a function of time for constant $e = 0.1$

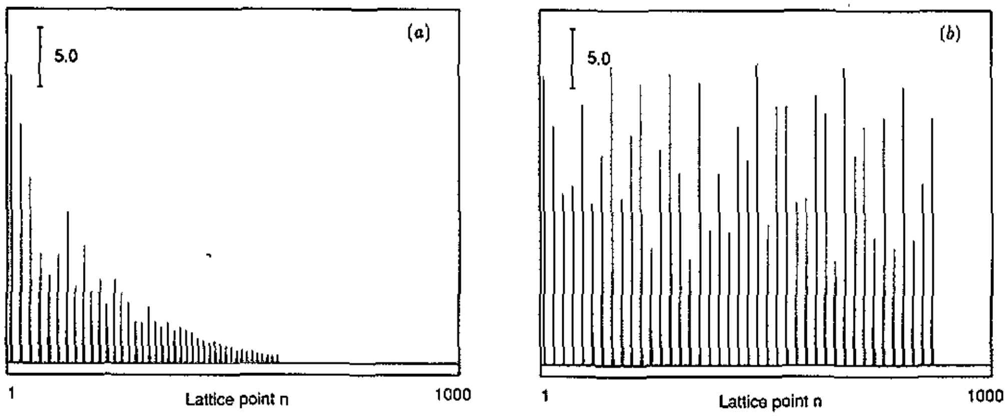

Figure 2. The soliton solution as a function of the lattice points at different moments of time $t = 0, 10, 20, \ldots, 400$. The initial amplitude was 24.37 and the dissipation factor $a = 0.01$. (*a*) The drive parameter $e = 0.23$ (below the threshold value). (*b*) $e = 0.24$ (above the threshold value).

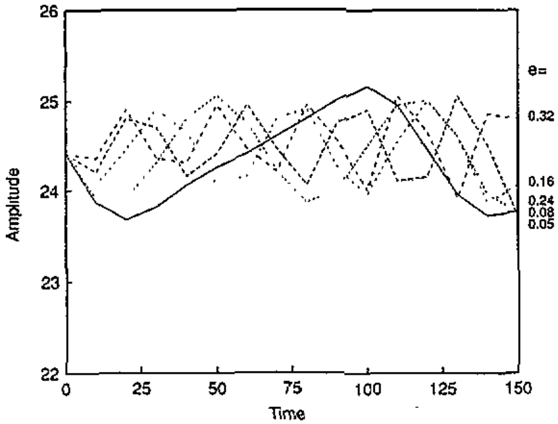

Figure 3. The amplitude of the soliton as a function of time with different values of the drive parameter above the threshold value ($A = 24.37$, $a = 0.002$).

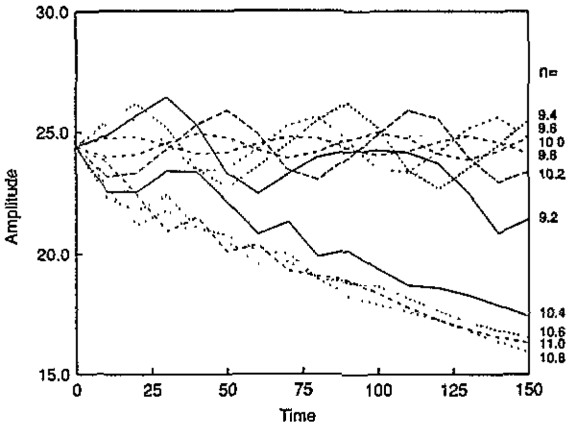

Figure 4. The amplitude of the soliton as a function of time with different initial positions n_0 ($A = 24.37$, $a = 0.002$, $e = 0.1$).

($A = 24.27$) but different initial position of the soliton. Obviously the case $n_0 = 10.0$ gives best results, i.e. the smallest variation in the amplitude. This means that the initial soliton should be positioned exactly on an even lattice point to create the best resonance drive. If n_0 is too far from the optimum value the resonance effect will never take place.

In figure 5 we have collected the threshold data for three different dissipation factors $a = 0.0005$, $a = 0.002$ and $a = 0.01$, each one with five different velocities corresponding $\omega = 1.0$, $\omega = 1.5$, $\omega = 2.0$, $\omega = 2.3$ and $\omega = 2.5$. The drive parameter is normalized with the dissipation factor in order to compare directly the theoretical prediction (4) with numerical results. With large amplitudes all curves converge and approach the theoretical curve. The result (4) is derived by assuming no radiative losses. Therefore the theoretical prediction for the threshold value is always smaller than the actual one needed for stable propagation of the soliton.

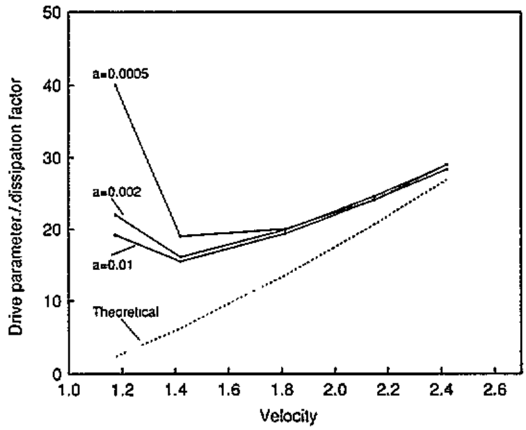

Figure 5. The threshold values of the drive parameter as a function of the velocity with different values of the dissipation factor. Threshold values are normalized with the dissipation factor. The solid lines represent numerical results and the dashed line the theoretical prediction ($=4\pi v \ln(v)$).

In the limit of the minimum velocity ($=1$) the threshold value tends to increase rapidly. This is because of the width of the soliton: when the soliton extends over several lattice points and the drive partially cancels itself. We also observe that with small dissipation a relatively stronger drive is needed. Indeed, as long as perturbation theory is applicable the rate of energy losses contains two parts [2], the dissipative part proportional to a and the radiative part independent of a:

$$e_{\text{thr}} = e_0 + ae_1 \tag{5}$$

and therefore the normalized drive parameter $\hat{e}_{\text{thr}} = e_0/a + e_1$ approaches infinity when a goes to zero, as figure 5 suggests.

The numerical results presented here show clearly that soliton dissipation can be compensated in a discrete lattice with an AC-drive which has alternating sign in adjacent lattice points. One can expect that this numerically observed effect can be seen in real solitons propagating in an AC-driven quasi-one-dimensional ionic lattice or in an ion doped polymer chain. Thus the effect described here should again provide further applications of the soliton concept in real systems.

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